Berkeley Math Circle IntermediateII **Factoradic and Modification**

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Nov. 29 and Dec. 6, 2023

1. If it has been 51 weeks, 6 days, 23 hours, 59 minutes, and 59 seconds since the start of the year 2023, how much longer till the start of the year 2024? (There are no leap seconds in 2023 and it is not a leap year.)

2. "Factorial base" (factoradic), or factorial number system

1) Notation

2) conversion and Reverse conversion from factorial base to base-10

3) Some arithmetic in factorial base

3. Problem: Every positive integer k has a unique factorial base expansion $(f_1, f_2, ..., f_m)$ meaning that $k = 1! \cdot f_1 + 2! \cdot f_2 + ... + m! \cdot f_m$ where each f_i is an integer, $0 \leq f_i \leq i$ and $0 < f_m$. Given that $(f_1, f_2, ..., f_j)$ is the factorial base expansion of $16! - 32! + 48! - 64! + ... + 1968! - 1984! + 2000!$, find the value of $f_1 - f_2 + f_3 - f_4 + ... + (-1)^{j+1} f_j$

4. Lexicographic order

4) E.g. $n = 3$

5) a) For $n = 6$:

i) rank(625143)= ?

ii) 300^{th} permutation =?

5) b) Finding successor and predecessor: For $n = 7$:

i) permutation after (653741)?

ii) Before (517346)?

6)Lehmer code:

(a)If $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ then $L(\sigma)_i = \#\ \{j > i : \sigma_j < \sigma_i\}$
So we get $L(\sigma) = (L(\sigma_1), L(\sigma_2), ..., L(\sigma_n))$

(b)(i, j) with $i < j$, and $\sigma_i > \sigma_j$ is called an inversion of σ.

(c) Notice that $L(\sigma)_i$ = inversions of (i, j) for a fixed i and varying j.

(d) $L(\sigma_1) + L(\sigma_2) + ... + L(\sigma_n)$ = total inversions of σ

(e) This is = # of adjacent transpositions to go to Identity permutation $(1, 2, ..., n)$.

5.

a) How about using digits $\{-1, 0, 1\}$ for Base$_3$? Digits $\{4, 5, 6\}$?

b) What are the valid sets of digits for Base$_3$? How about for Base$_4$?

c) Compare: Size of Base$_n$ intervals between n^k and n^{k+1} increases monotonically as a function of k. Similar pattern for factorial base.

d) Contrast: Number of digits stays constant, n, for Base$_n$, but increases by 1 for each consecutive place value.

6. Modification of the factoradic.

Problem 1): (AMC 10 B 2022 # 9): The sum $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{2021}{2022!}$ can be expressed as $a - \frac{1}{b!}$, where a, b are positive integers. What is $a + b$?

Problem 2): Prove that any positive integer less than $n!$ can be represented as a sum of no more than n positive integer dividers of $n!$.

7. Decreasing function f : [k]–> [n]

(i) List decreasing function f : [3]–> [5] in lex order.

(ii) rank (5 4 2) = ? (Draw the diagram)

(iii) For f : [4]–> [12], rank(10, 7, 5, 2) = ? (Calculate using formula.)

(iv) For f : [3]–> [9], unrank(42) = ?

8. Practice problems:

1) If a language has only four alphabets A, K, N, R in that order giving rank$(AKNR) = 0$, what is the rank$(RANK)$?

2) Write 2982_{10} in factorial number system.

3) What is the largest "7-length" long number in factorial system? (It has 6 colons.) Write it as a factorial number and also find its value in decimal system.

4) What is the next number in the factorial system? I.e. What is the smallest "8-length" long number in factorial system?

5) What is the successor of (4761532)?

Problems from AMC, AIME, USA IMO Team Training.
Material from many online open sources and (UCSD Online document from Prof. Tesler's course)